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The Emergence of Coherent Structures in Coupled Map Lattices

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We show how increasing spatial interaction leads to the merging of coherent structures from chaos in some systems of coupled map lattices. This phenomenon reflects the arising of new ground states in the corresponding model of statistical mechanics. If we further increase the coupling then, new ground states appear showing the coexistence of a large-scale coherent structure with a small-scale chaotic motion. This allows us to propose a generalization of the notion of spatial intermittency.

KEY WORDS: Coupled map lattices; coherent structures; chaos; spatial intermittency.

1. Coupled map lattices (CML) were recently introduced in order to study spatially extended dynamical systems and are now among the most popular objects in nonlinear dynamics.[1,2]

These systems give us the possibility to study, in a very simple setting, different phenomena, for instance, space-time intermittency, spatial bifurcation (e.g., as it appears in open flows[2]), and spatial patterns of different types.[3]

From a general point of view one of the main problems is certainly the construction of the statistical mechanics of systems presenting spatiotemporal complexity, e.g., systems with a large number of excited modes. This approach, the so-called thermodynamic formalism, appears to be one of the most effective tools to study the statistical properties in the theory of

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hyperbolic (chaotic) dynamical systems with a finite number of degrees of freedom.[4] This method was applied[1] to some CML with diffusion type coupling, i.e., to dynamical systems with infinite degrees of freedom. It was proved in ref. 1 that the dynamical system generated by space translation and the dynamics has a unique invariant Gibbs measure, provided the coupling is small enough. It is also proved in ref. 1 that, with respect to that measure, the space and time correlations decay to zero. This property corresponds to the absence of phase transition for the corresponding lattice model of statistical mechanics at high temperature.

The hyperbolic dynamical systems which are not spatially distributed correspond to one-dimensional statistical mechanics lattice models with rapidly decaying interactions,[4] and therefore they do not show phase transitions. If we consider CML, then the situation radically changes. These dynamical systems have a symbolic representation by a statistical dynamical model with at least two dimensions,[4] the space dimension plus the dynamical one. Their symbolic representation by a statistical mechanical model has at least two dimensions.[4] Therefore, if ref. 1, the conjecture was made that by increasing the space interaction in CML, the system can undergo a phase transition.

In this context, the appearence of new ground states can be interpreted as the arising of coherent structures from chaos in systems with an infinite number of degrees of freedom. In some cases, it is also possible to give a reasonable definition of the coherent structures using this interpretation.

In the present paper we shall discuss the results of our numerical experiments giving the evidence for such a conjecture. We have also carried out the corresponding analytic proof of these results giving explicit formulas for the critical values of the parameters in each case. We will analyze the situation from this point of view in a future article.[5]

2. Here, we deal with a lattice of N maps f of the unit interval into itself, which are expanding or quadratic. Given the state of the system at time n, $x^n=(x_i^{(n)})$, $1 \leqslant i \leqslant N$, the new state at the time $n+1$ is calculated according to the following formula:

$$x_i^{(n+1)} = (1-\varepsilon) f(x_i^{(n)}) + \frac{\varepsilon}{2} [f(x_{i-1}^{(n)}) + f(x_{i+1}^{(n)})] \tag{1}$$

In the following we only show the case of periodic boundary conditions, as different boundary conditions do not affect the results in any essential way.

Let us first consider the case of an f with very strong chaotic properties, a so-called expanding map, that is, a function f which is pointwise smooth with a finite number of singularities only and $|f'(x)| > \lambda > 1$ for any nonsingular point x in the unit interval (for singular points the same inequality is required for left and right derivatives).

In Fig. 2, we show the result of our numerical computations for the following expanding map (similar cases were treated with essentially the same results): f is linear in each interval $[i/10, i+1/10]$, $i=0, 1,..., 9$, the (see Fig. 1) with

$$\begin{array}{lllll} f(0)=\frac{1}{5}, & f(\frac{1}{10})=\frac{2}{5}, & f_-(\frac{2}{10})=\frac{1}{5}, & f_+(\frac{2}{10})=\frac{3}{5}, & f(\frac{3}{10})=\frac{4}{5} \\ f_-(\frac{4}{10})=\frac{3}{5}, & f_+(\frac{4}{10})=\frac{4}{5}, & f(\frac{5}{10})=\frac{3}{5}, & f_-(\frac{6}{10})=\frac{4}{5}, & f_+(\frac{6}{10})=\frac{1}{5} \\ f(\frac{7}{10})=\frac{2}{5}, & f_-(\frac{8}{10})=\frac{1}{5}, & f_+(\frac{8}{10})=\frac{3}{5}, & f(\frac{9}{10})=\frac{4}{5}, & f(1)=\frac{3}{5} \end{array} \tag{2}$$

where f_- (resp. f_+) stands for the limit from the left (resp. the right) of f.

In Fig. 2a the ground state for small ε is shown, which is unique according to the result proved in ref. 1. Notice that this map is defined in such a way that the intervals $I_1=[1/5, 2/5]$ and $I_2=[3/5, 4/5]$ are mapped one in the other and, after a transient of one iteration, all the points of $[0, 1]$ end up in $I_1 \cup I_2$. Therefore, for ε sufficient small, according to ref. 1, there is only one absolutely continuous invariant measure with support in $I_1 \cup I_2$. The induced measure is mixing with respect to space as well as time translations.

In Fig. 2a we report a typical configuration for such values of ε. As for $\varepsilon=0$, each point of the lattice jumps from I_1 to I_2 or vice versa at each time iteration. According to the initial conditions we use (see Figs. 2 and 3), half of the points (corresponding, say, to odd sites) jump from I_1 to I_2 and at

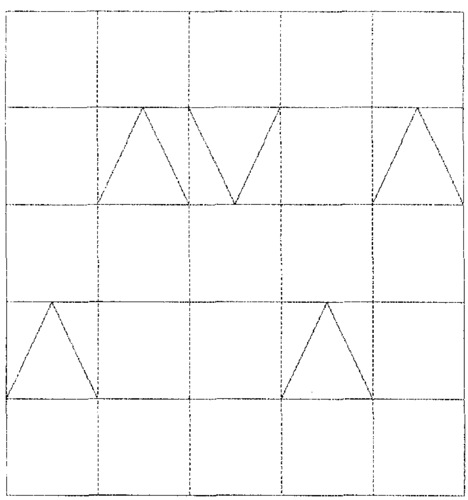


Fig. 1. The map f defined in (2).

the same time the other half (even sites) jump from I_2 to I_1. After two iterations all the sites return to their initial interval.

Notice also that, for sites sufficiently distant from one another, the corresponding values of the variables are almost independent; this fact has been tested by correlation decay, but we do not present here the corre-

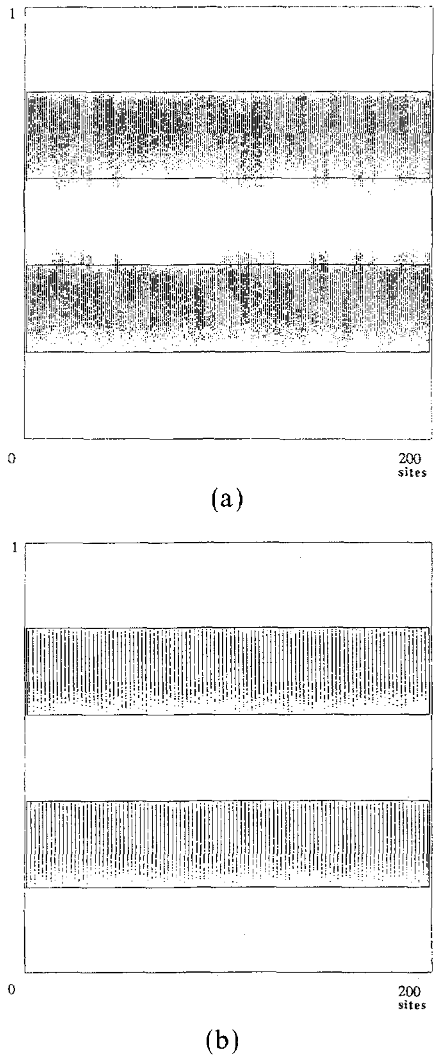
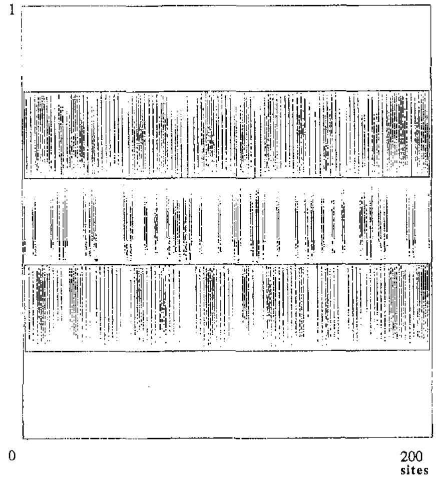


Fig. 2. Plot of about 200 iterations after a transient of 100 iterations for the expanding map (2) (horizontal axis, $0 \leqslant i \leqslant 200$; vertical axis, $0 \leqslant x \leqslant 1$): (a) $\varepsilon = 0.2$ with initial random conditions for even points in I_1 and odd points in I_2. (b) $\varepsilon = 1.0$ with initial conditions as in (a). (c) $\varepsilon = 1.0$ with initial random conditions in $I_1 \cup I_2$.



(c)

Fig. 2. *(Continued)*

sponding data, since they are of the same type as those of the map treated below.

Upon increasing ε, the system performs a bifurcation, and two new stable solutions appear in the phase space of the system. As it is shown in Fig. 2b, these correspond to the situation where all sites $x_i^{(n)}$ for all odd i's (resp. even) move only inside the interval I_1 (resp. I_2). Therefore this bifurcation can be considered as giving rise to a phase synchronization of the motion in space.

But the chaotic ground state still survives (Fig. 2c) for other types of initial conditions.

Figure 3 gives the result of the corresponding numerical experiments for the logistic quadratic map:

$$f(x) = ax(1-x) \tag{3}$$

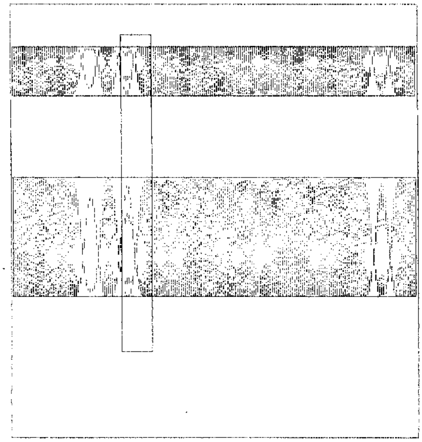
with boundary conditions as in the previous case.

We take the value of the parameter a ($a = 3.6...$) corresponding to a situation for which it is known[6] that the only invariant measure for f is concentrated in two disjoint subintervals I_1 and I_2 of the unit interval.

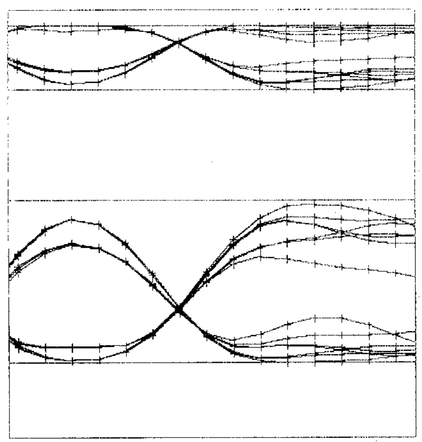
Again, for small ε there is only one ground state and, as shown in Fig. 3a, this corresponds, as in the previous case, to the jumping of points, first from I_1 to I_2 and then from I_2 to I_1 in all sites of the lattice. Notice the important fact that here the position of all the points is such that their projections fill each of the two intervals (see also Fig. 3b).

When ε is increased, a bifurcation takes place at $\varepsilon \cong 0.884$ giving rise to two new stable solutions. The corresponding states of the lattice are standing waves in space with period two, when all even points of the lattice are fixed at one level and all the odd points at another level (see Fig. 3c).

These standing waves are the simplest stable coherent structures in

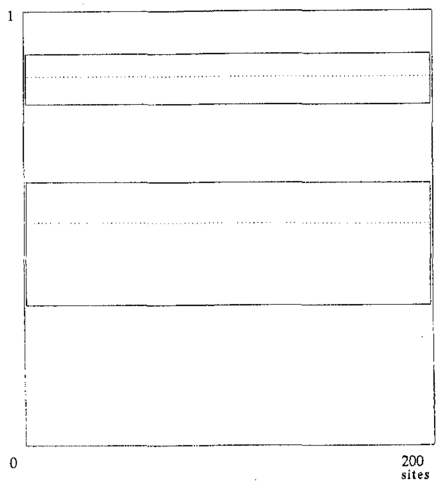


(a)

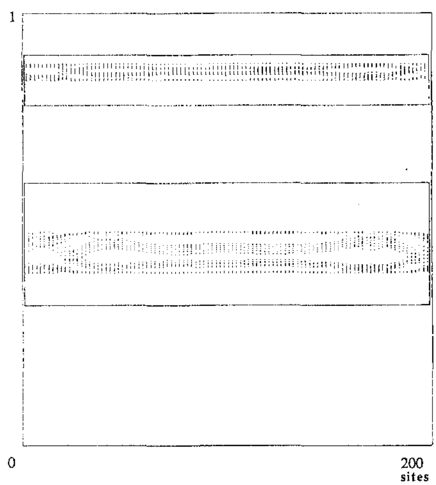


(b)

Fig. 3. Plot of about 200 iterations after a transient of 100 iterations for the quadratic map, (horizontal axis, $0 \leqslant i \leqslant 200$; vertical axis, $0 \leqslant x \leqslant 1$): (a) $\varepsilon = 0.88$ with initial random conditions for even points in I_1 and odd points in I_2. (b) Magnification of 15 sites of (a) for 25 iterations running after the last iteration of (a). (c) $\varepsilon = 0.884$ with initial conditions as in (a). (d) $\varepsilon = 0.96$ with initial conditions as in (a) (only 20 iterations).



(c)



(d)

Fig. 3. (*Continued*)

space. We are therefore clearly in the presence of the emergence of a coherent structure from chaos.

Upon increasing ε again, the corresponding ground states are preserved as long as ε does not exceed 0.95. At this value, a new bifurcation takes place and, instead of the two standing waves described before, two other ground states appear. For these values of ε, half of the points stay in I_1, whereas the remaining points stay in I_2, but they are no longer fixed.

This corresponds to the coexistence of large-scale standing waves with small-scale random motion of the points inside two small intervals, $J_1(\varepsilon) \subset I_1$ and $J_2(\varepsilon) \subset I_2$ (see Fig. 3d). Notice that the size of these intervals goes to zero as ε decreases to the critical value.

It is therefore natural to consider these ground states as the simplest qualitative model for the small-scale turbulent motion inside a motion which is regular in a larger scale. Such behavior is apparent in many hydrodynamic systems.[7]

Let us also mention that, for the value of the parameter a for which it is known that the invariant measure of the one-dimensional map is concentrated in four disjoint intervals, there are also ground states corresponding to moving waves of (space) period four.

Let us now come to the problem of the intermittency in spatially distributed systems. Usually one means by this that there exists some partition of the physical space of the system so that in some of the domains (the elements of the partition) the motion is chaotic and in others it is regular; furthermore, it is generally assumed that these domains have more or less sharp boundaries.

Instead, if we have in mind the previous analysis and the fact that, at least in some cases, a spatially distributed dynamical system can be represented as a statistical mechanical system, and that this system can have several ground states, then clearly a more general picture (or notion) of space intermittency emerges. In this case, if we take random initial condition (the initial configuration of the lattice), then the system can be divided at initial time into different parts that (locally) are typical for different ground states. The type of motion in each of these domains of the lattice will be different. We are therefore in the presence of a regime (at least a transient one) in which not only chaotic and regular motion coexist in different domains, but also different types of chaotic motion can be present. Such flows have been recognized in many hydrodynamic experiments;[7] especially for Rayleigh–Bénard convection in the annulus.[8]

In Fig. 4 we give an example of such an intermittency taken from our numerical experiment with the second model, in which we can see different regimes occupying regions with a typical intermediate length scale, separated by bursts in randomly located sites.

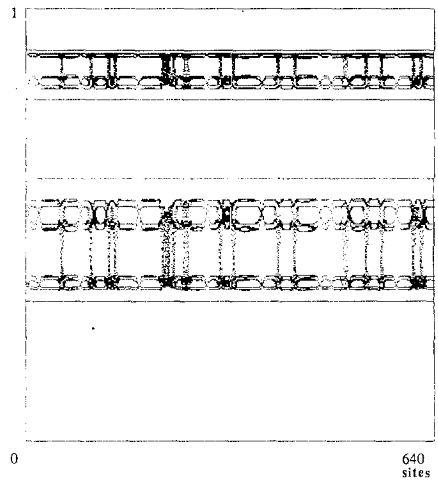
The corresponding normalized spatial correlation function defined as

$$F(q) = \frac{\sum_j x_i^{(n)} x_{i+2q}^{(n)} - \langle x^{(n)} \rangle_i^2}{\langle (x^{(n)})^2 \rangle_i - \langle x^{(n)} \rangle_i^2} \tag{4}$$

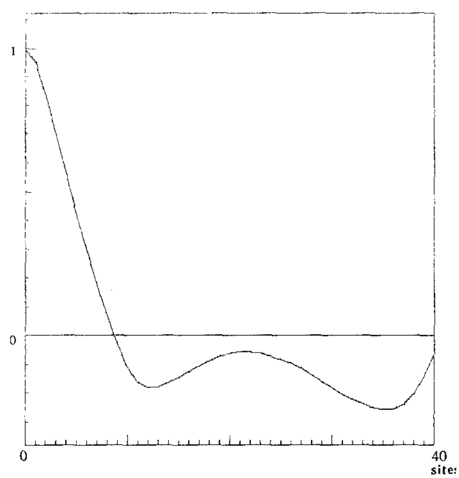
where

$$\langle u \rangle_i = \frac{1}{N} \sum_i u_i$$

is represented in Fig. 4b for the same values of the parameters and initial conditions. It shows the independence of the corresponding motion for distant sites, even if we are careful in the interpretation of a finite-size numerical experiment, since different types of decay of correlations (not only exponentials) are expected in the infinite-volume limit.



(a)



(b)

Fig. 4. (a) $\varepsilon = 0.985$, initial conditions as in Fig. 2, with 300 configurations for the quadratic map, for a lattice of 640 sites. (b) $\varepsilon = 0.985$, space correlation function for even points of the lattice up to a distance of 45 sites.

It can be seen that the structure of this state is very similar to that obtained in the experimental situation studied in ref. 8.

3. In this paper we have shown how increasing spatial interaction leads to the merging of coherent structures in the configuration space of a system of coupled map lattices.

The new phenomenon reported here is the following: starting from a turbulent state, which is known to exist and to be unique for small coupling of this system, simpler structures appear corresponding to a more organized type of motion when the coupling is increased. These simpler structures can then be considered as coherent structures for this system.

Another relevant property of the system is the existence of windows in the coupling parameter for the appearence of each type of coherent structure. Therefore, this phenomenon is not simple freezing due to the large value of the coupling, but a special situation where a new ground state appears in the system for a given range of the coupling parameter.

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